VALUATIONS WITH GIVEN RESIDUE CLASS FIELD AND VALUE GROUP

BY

PHILIP WILKINSON CARRUTH

A.B., Hamilton College, 1936

A.M., Syracuse University, 1937

AN ABSTRACT OF A THESIS

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN MATHEMATICS IN THE GRADUATE SCHOOL OF THE UNIVERSITY OF ILLINOIS, 1941

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INTRODUCTION

Associated with every field with a valuation is a certain field of residue classes and a value group. But corresponding to a given arbitrary field and an arbitrary ordered abelian group, there exist many fields with valuations that have the given field as a residue class field and the given group as a value group. It is the purpose of this paper to investigate fields with valuations that have the same residue class field and value group. We shall look into the "degrees of perfection" of such fields, and we shall give criteria for such fields to be maximal. Our investigations along this line will be carried out in Part II.

In Part I, we shall prove several lemmas on the arithmetic of ordinals, which give upper bounds to the order types of certain ordered sets. These lemmas have some intrinsic interest and also are needed for our investigations in Part II.

Part I: ARITHMETIC OF ORDINALS WITH APPLICATIONS TO THE THEORY OF ORDERED ABELIAN GROUPS.

Addition and multiplication of ordinals have been defined in a logical way. However, these operations do not behave as well as one might desire. For example, the commutative laws are not valid, and the distributive law is valid on only one side. Consequently we make some definitions.

Definition I.1: $\alpha \oplus \beta$ is termed a natural sum of α and β if $\alpha \oplus \beta$ is a well-determined ordinal for any two ordinals α and β, and if:

1. $\alpha \oplus \beta = \beta \oplus \alpha$
2. $(\alpha \oplus \beta) \oplus \delta = \alpha \oplus (\beta \oplus \delta)$
3. $\alpha \oplus 0 = \alpha$
4. $\delta \oplus \alpha > \delta \oplus \beta$ if and only if $\alpha > \beta$,

where δ is any ordinal.

Throughout this abstract, $\nabla(\alpha, \beta)$ will denote the particular natural sum defined by Hessenberg[1]. It is the unique natural sum satisfying the condition that $\omega^{\alpha} \cdot m + \omega^{\beta} \cdot n = \nabla(\omega^{\alpha} \cdot m, \omega^{\beta} \cdot n)$, where α and β are any two ordinals such that $\alpha \geq \beta$, and where m and n are any two positive integers.

We prove

Theorem I.2: I Let T and R be well-ordered subsets, of an ordered set of elements, of respective order types τ and ϱ. Then the set of elements in the join of R and T, i.e. R+T, is well-ordered and has order type $\leq \tau \oplus \varrho$, where $\tau \oplus \varrho$ is any natural sum of τ and ϱ.

II There exists a well-ordered subset R, of order type ϱ, of an ordered set, and a well-ordered subset T, of order type τ, of the same ordered set, such that the order type of R+T is exactly $\nabla(\varrho, \tau)$.

III $\nabla(\varrho, \tau) \leq \tau \oplus \varrho$, where $\tau \oplus \varrho$ is any natural sum.

IV $\tau + \varrho \leq \tau \oplus \varrho$, where $\tau \oplus \varrho$ is any natural sum.

Definition I.3: $\alpha \otimes \beta$ is termed a natural product of α and β if $\alpha \otimes \beta$ is a well-determined ordinal for any two ordinals α and β, and if:

1. $\alpha \otimes \beta = \beta \otimes \alpha$
2. $(\alpha \otimes \beta) \otimes \delta = \alpha \otimes (\beta \otimes \delta)$
3. $\alpha \otimes 1 = \alpha$
4. $\alpha \otimes \delta > \beta \otimes \delta$ if and only if $\alpha > \beta$
5. $\nabla(\alpha \otimes \beta, \alpha \otimes \delta) = \alpha \otimes \nabla(\beta, \delta)$
6. $\omega^{\alpha} \otimes \omega^{\beta} = \omega^{\gamma}$

where δ is any ordinal, and where $\gamma = \gamma(\alpha, \beta)$ is a suitable ordinal.

Throughout this abstract, $\Pi(\alpha, \beta)$ will denote the particular natural product defined by Hausdorff[2]. It is obtained by imposing the condition that the ordinal γ in 6) of I.3 be equal to $\nabla(\alpha, \beta)$.

Definition I.4: An ordered set of elements, $A = \{a_{\alpha\beta}\}$, $0 \leq \alpha < \tau$, $0 \leq \beta < \rho$, where τ and ρ are any two ordinals, is an ordered τ, ρ-block if $a_{\alpha\beta} < a_{\alpha\gamma}$ and $a_{\beta\alpha} < a_{\gamma\alpha}$ for $\beta < \gamma$.

We now prove

1) G. Hessenberg, "Grundbegriffe der Mengenlehre", Abhandlungen der Fries'schen Schule, New Series 1.4, Göttingen, 1906, #75.

2) F. Hausdorff, "Mengenlehre", Third Edition, p. 70.

Theorem I.5: I Let $A=\{a_{\alpha\beta}\}$ be an ordered τ,μ-block. Then A is well-ordered and has order type $\leq \tau \otimes \mu$, where $\tau \otimes \mu$ is any natural product.
II There exists an ordered τ,μ-block of order type exactly $\Pi(\tau,\mu)$.
III $\Pi(\tau,\mu) \leq \tau \otimes \mu$, where $\tau \otimes \mu$ is any natural product.
IV $\tau \cdot \mu \leq \tau \otimes \mu$, where $\tau \otimes \mu$ is any natural product.

We conclude this part by proving

Theorem I.6: Let G be an ordered abelian group. Let T be a well-ordered (according to size) set of positive elements in G of order type $\beta = \tau + n$, where τ is a limit ordinal or 0 and n is an integer greater than or equal to 0. Let M be the set of all finite sums of elements in T. Then M is well-ordered and has order type $\leq \Pi(\omega^{\beta}, \tau^{\tau})$.

Part II: FIELDS WITH A VALUATION.

A function V of the elements in a field F is a valuation of F with a value group G, where G is an ordered abelian group, if for all $a \neq 0$, $b \neq 0$ in F:

1. $V(a)$ is an element in G.
2. $V(ab) = V(a) + V(b)$
3. $V(a+b) \geq \text{minimum}\{V(a), V(b)\}$

Let F be a field with a valuation V. The set R of all a in F such that $V(a) \geq 0$ forms a domain of integrity whose quotient field is F. The set P of all a in F such that $V(a) > 0$ forms a largest prime ideal in R. The field $K = R$ modulo P is called the residue class field of F with respect to the valuation V.

Throughout this abstract, F will denote a (K,G) field, i.e. a field with a valuation with residue class field K and value group G, where K is an arbitrary field and G is an arbitrary ordered abelian group.

Let F be a subfield of a field H that has a valuation V'. The valuation V' of H is said to be an extension of the valuation V of F if $V'(a) = V(a)$ for every element a in F.

F is said to be maximal if there exists no field H containing F as a subfield such that:

1. There is a valuation V' of H that is an extension of the valuation V of F such that $V'(a)$ is in G for every a in H.
2. Corresponding to every element a in H such that $V'(a) = 0$, there is an element b in F such that $V'(a - b) > 0$.

Definition II.1: The sequence $\{a_{\alpha}\}$, $0 \leq \alpha < \tau$, where τ is a limit-ordinal, of elements in F is a pseudo-convergent sequence if

1. $V(a_{\alpha}) = 0$ for all α.
2. If $\beta > \mu$, $\gamma > \mu$, $\delta > \mu$, then $V(a_{\mu} - a_{\beta}) < V(a_{\gamma} - a_{\delta})$.

An element b in F is a pseudo-limit of the pseudo-convergent sequence $\{a_{\alpha}\}$ if for $\beta > \mu$, $V(b - a_{\mu}) \geq V(a_{\beta} - a_{\mu})$.

F is properly perfect if every pseudo-convergent sequence in F has a pseudo-limit in F.

Let τ be any limit-ordinal. Then F is τ-perfect if every pseudo-convergent sequence $\{a_{\alpha}\}$, $0 \leq \alpha < \gamma$, $\gamma \leq \tau$, in F, has a pseudo-limit in F.

Let $B_0, B_1, B_2, \ldots$ be the well-ordered set of all ordinal number classes. Let A be the set of all order types of all proper monotone increasing well-ordered (according to size) sequences of elements in G. Let $B(G) = B_{\varrho}$ be the first class such that every ordinal of A is in a class B_{α}, $\alpha \leq \varrho$. Let $\alpha(G)$ be the first ordinal in the class B(G). Then we prove

Theorem II.2[3]: The following three statements are equivalent:

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[3] Through correspondence, since completing the proof of this theorem, the writer has learned that Mr. Irving Kaplansky has submitted a paper on Maximal Fields with Valuations to the American Mathematical Society in which is contained a proof of the equivalence of I and III. For methods of proof similar to those used by the writer, see: A. Ostrowski, "Untersuchungen zur Arithmetischen Theorie der Körper", Mathematische Zeitschrift, Vol. 39 (1934-35), p. 370-375; W. Krull, "Allgemeine Bewertungstheorie", Journal für die Reine und Angewandte Mathematik, Vols. 166-167 (1931-32), (hereafter referred to as 'Krull'), p. 176-180, p. 194-195.

I F is properly perfect.
II F is $\alpha(G)$-perfect.
III F is maximal.

Corollary II.3: If G is an additive group of real numbers, then F is maximal if and only if F is ω-perfect.

Definition II. 4: Let S be the generalized power series field constructed by Hahn[4], consisting of all formal power series, $\sum_{0\le\alpha<\tau} a_\alpha x^{g_\alpha}$, where the a_α are in K, τ is any ordinal, and the g_α form a proper monotone increasing sequence of elements in G.

We now prove

Theorem II.5:[5] If F is maximal, then the cardinal number of F is the same as the cardinal number of S.

Definition II.6: Let $\overline{F}$ be a system of representatives in F of the residue classes in K. If a is in F such that $V(a)\ge 0$, then let $\overline{a}$ be the element in $\overline{F}$ that belongs to the same residue class as a. Let $\overline{0}=0$ and $\overline{1}=1$. Let $\overline{G}$ be a system of representatives in F of the values in G. $\overline{G}=\{\Pi_g\}$, where $V(\Pi_g)=g$. Let $\Pi_0=1$. If a is in F and $V(a)\ge 0$, then $a-\overline{a}=H(a)$.

We now show that we may select a unique set C of elements in F and a unique set L of elements in F such that

1. If c is in C, then $V(c)=0$. If x is in L, then $V(x)>0$.
2. $H(\sum_{i=1}^{n} a_i b_i d_i + \sum_{i=1}^{m} c_i)$ is in L for all a_i, b_i in $\overline{F}$, all c_i, d_i in C, and for every pair of integers $n, m \ge 0$.
3. If x is in L and $x = c(x)\Pi_{V(x)}$, then c(x) is in C.
4. $\Pi_g \cdot \Pi_h = c(g,h)\Pi_{g+h}$ implies that c(g,h) is in C.
5. If c_1 and c_2 are in C, then $c_1 \cdot c_2$ is in C.
6. If x is in L and $x\cdot\Pi_g = c(x,g)\Pi_{V(x)+g}$, then c(x,g) is in C.

We impose the following

Condition II.7: The set T of all the values of the elements in L is well-ordered according to size. Let $A=\sum_{i=1}^{n} a_i\Pi_{g_i}$, and $B=\sum_{i=1}^{m} b_i\Pi_{g_i}$, where the a_i and b_i are in $\overline{F}$, and n and m are integers greater than 0. Then $A+B=\sum_{i=1}^{r} h_i\Pi_{v_i}$ and $AB=\sum_{i=1}^{k} s_i\Pi_{w_i}$, for suitable h_i and s_i in $\overline{F}$, v_i and w_i in G, and integers $r, k > 0$.

Definition II.8: Let $H=H(\overline{F},\overline{G})$ be the set of all formal power series, $\sum_{0\le\alpha<\nu} a_{g_\alpha}\Pi_{g_\alpha}$ where ν is any ordinal, the a_{g_α} are in $\overline{F}$, and the g_α are well-ordered according to size.

Definition II.9: M is the set of all finite sums of elements in T.

II.10 Addition in $H(\overline{F},\overline{G})$

Let $A=\sum_{0\le\alpha<\nu} a_{g_\alpha}\Pi_{g_\alpha}$ and $B=\sum_{0\le\alpha<\nu} b_{g_\alpha}\Pi_{g_\alpha}$ be any two elements in H.

$A+B=D=\sum_{0\le\alpha<\mu} d_{g_\varepsilon}\Pi_{g_\varepsilon}$, where the f_ε are the set $\{g_\alpha+M\}$, i.e. $\{g_\alpha+m_\beta\}$ for $0\le\alpha<\nu$, and for all m_β in M. If there is no g_α such that $g_\alpha=f_\varepsilon$, then let $a_{g_\varepsilon}=b_{g_\varepsilon}=0$. $d_{g_0}=\overline{a_{g_0}+b_{g_0}}$, and $H_{g_0}=H(a_{g_0}+b_{g_0})$. Assume that all d_{g_α} have been defined for $\alpha<\tau$, and every H_{g_α} determined for $\alpha<\tau$.

For $\alpha<\tau$, let $H_{g_\alpha}\Pi_{g_\alpha}=c_{g_\alpha}\Pi_{V(H_{g_\alpha})+g_\alpha}$.

Then $d_{g_\tau}=\overline{a_{g_\tau}+b_{g_\tau}+\sum_{V(H_{g_\alpha})+g_\alpha=g_\tau} c_{g_\alpha}}$.

$H_{g_\tau}=H(a_{g_\tau}+b_{g_\tau}+\sum_{V(H_{g_\alpha})+g_\alpha=g_\tau} c_{g_\alpha})$, and $H_{g_\tau}\Pi_{g_\tau}=c_{g_\tau}\Pi_{V(H_{g_\tau})+g_\tau}$

II.11 Multiplication in $H(\overline{F},\overline{G})$

Let $A=\sum a_{g_\alpha}\Pi_{g_\alpha}$ and $B=\sum b_{g_\alpha}\Pi_{g_\alpha}$ be any two elements in H.

$AB=D=\sum d_{g_\varepsilon}\Pi_{g_\varepsilon}$, where the f_ε consist of the set $\{g_\alpha+g_\beta+M\}$.

$d_{g_0}=\overline{a_{g_0}\cdot b_{g_0}\cdot c(g_0,g_0)}$, and $H_{g_0}=H(a_{g_0}\cdot b_{g_0}\cdot c(g_0,g_0))$. Assume that all d_{g_α} have been defined for $\alpha<\tau$, and every H_{g_α} determined for $\alpha<\tau$. For $\alpha<\tau$,

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4) H. Hahn, "Nichtarchimedische Grössensysteme", Wien K. Akademie der Wissenscheften, Sitzungsberichte, Math.-Nat. Classe, Vol. 116, 2A, (1907), p. 647-655.

5) In the proof of this theorem, the writer used a map of F into S constructed by Krull, p. 192-193.

let $H_{g_\alpha}\Pi_{g_\alpha} = c_{g_\alpha}\Pi_{V(Hg_\alpha)+g_\alpha}$. Then $d_{g_\tau} = \sum_{g+h=g_\tau} a_g b_h c(g,h) + \sum_{V(Hg_\alpha)+g_\alpha=g_\tau} c_{g_\alpha}$

$H_{g_\tau} = H(\sum_{g+h=g_\tau} a_g b_h c(g,h) + \sum_{V(Hg_\alpha)+g_\alpha=g_\tau} c_{g_\alpha})$, and $H_{g_\tau}\Pi_{g_\tau} = c_{g_\tau}\Pi_{V(Hg_\tau)+g_\tau}$

We now prove

Theorem II.12: I With addition and multiplication defined as in II.10 and II.11, $H(\bar{F},\bar{G})$ forms a field.

II Let $A = \sum a_{g_\alpha}\Pi_{g_\alpha}$ be any element in H, $a_{g_0} \neq 0$. Let $V(A) = g_0$. Then the function V constitutes a valuation of H.

III With respect to the valuation V of II, H is a properly perfect (and so maximal) (K,G) field.

We give two examples of fields $H(\bar{F},\bar{G})$. One of these turns out to be the field S of II.4. The other is a certain infinite extension of a p-adic field, and the valuation of it has the additive group of all rational numbers as its value group.

A sequence $\{a_\alpha\}$, $0 \leq \alpha < \tau$, of elements in F is said to be a regular sequence if corresponding to any element g in G, there is an ordinal $\rho_g < \tau$ such that $V(a_\alpha - a_\beta) > g$ for all $\alpha, \beta > \rho_g$. An element b in F is the limit of the regular sequence $\{a_\alpha\}$, if corresponding to any element g in G, there is an ordinal $\lambda_g < \tau$ such that $V(b - a_\alpha) > g$ for all $\alpha > \lambda_g$.

Theorem II.13: Let μ be the order type of the set M of II.9. Let τ be the smallest ordinal such that every proper monotone increasing well-ordered (according to size) sequence of elements in G is of order type $\leq \tau$. If G is archimedian, let γ be any ordinal that is greater than any finite product of ordinals that precede it, such that $\mu \leq \gamma < \tau$. If G is non-archimedian, let γ be any ordinal such that $\gamma = \omega^\gamma$ and such that $\mu < \gamma < \tau$. Let R_γ be the set of all elements $A = \sum a_{g_\alpha}\Pi_{g_\alpha}$ in $H(\bar{F},\bar{G})$ such that the order type of the g_α is $\leq \gamma$, the equality holding only if the g_α approach infinity. Then

I R_γ is a subfield of H.

II With respect to the valuation of R_γ induced by the valuation of H defined in II of II.12, R_γ is a (K,G) field.

III Every regular sequence in R_γ has a limit in R_γ.

IV Let β be the smallest ordinal such that there is a sequence of ordinals $\alpha_0 < \alpha_1 < \alpha_2 < \ldots.$ of order type β that approaches γ as a limit, each $\alpha_s < \gamma$. Then R_γ is ∇-perfect if and only if $\nabla < \beta$.

A field F with a valuation with an archimedian value group is said to be complete if every regular sequence in F has a limit in F.

We conclude this paper by proving

Theorem II.14: Let K be a field and G an archimedian ordered abelian group. Then there exists a complete (K,G) field that is not maximal, if and only if there exists a well-ordered (according to size) set of elements in G of order type greater than ω.

VITA

Philip Wilkinson Carruth was born July 19, 1914, in Cleveland, Ohio. He attended the public schools in Shaker Heights, Ohio, and graduated from Shaker Heights High School in 1932; attended Hamilton College from 1932-1936, where he received the degree of Bachelor of Arts in 1936; attended Syracuse University from 1936-1937, where he received the degree of Master of Arts in 1937. From 1937 to 1941, he was a Graduate Assistant at the University of Illinois.

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